Physics 250 Intro to Heavy Ions – Homework 1

Physics 250-4 Spring 2001

1. In the lab, two particles approach one another, each with a speed of $v = 0.8c$. In the lab frame, what is the relative velocity of the particles? If we hop to the frame of one of the particles, what is the velocity of the other?

2. In problem 1, we can't add the lab velocities to get the relative velocity in the frame of the particles. However, in one dimension $y = \tanh^{-1}(U)$, called the *rapidity*, *does* add for any velocity U (again, using $c = 1$). It acts "Galilean" and, in relativistic collisions, this proves to be quite useful. Show generally that, for one dimensional motion, the rapidities add under boosts. Calculate the specific rapidities in problem 1 and convince yourself that the additive property really works.

3. Because of the "longitudinal nature" of a typical high energy experiment, it is common to define the rapidity in terms of a particle's velocity along the beam direction (z) rather than along its own direction of motion. Show that for a particle moving with velocity $\vec{u}$ the z-component, u_z, can be written p_z/E where p_z is the longitudinal momentum and E is the total energy. Show that the rapidity can then be written as $y = \frac{1}{2}\ln(\frac{E+p_z}{E-p_z})$.

4. Show that the expressions for the total energy $E = \gamma m$ and $E = m_T \cosh y$ are equivalent. Here, $y = \tanh^{-1}(u_z)$ and $m_T = \sqrt{m^2 + p_T^2}$ are the rapidity and transverse mass respectively. The longitudinal velocity is u_z while p_T is the transverse momentum. The factor gamma is given by $\gamma = \frac{1}{\sqrt{1-u^2}}$ where u is the magnitude of the total velocity.

5. Show that the expressions $p_z = \gamma m u_z$ and $p_z = m_T \sinh y$ are equivalent. Here, $y = \tanh^{-1}(u_z)$ and $m_T = \sqrt{m^2 + p_T^2}$ are the rapidity and transverse mass respectively. The longitudinal velocity is u_z while p_T is the transverse momentum.

6. Show that although the transverse *velocity* does not remain invariant under longitudinal boosts, the transverse *momentum* does.

7. In problems 4 and 5, the transverse mass, $m_T = \sqrt{m^2 + p_T^2}$ was introduced. What is the physical interpretation of the transverse mass?

Homework 1 Solutions

1. In the lab frame, the relative velocity is as expected: $1.6c$. The distance between the two particles will shorten with this speed on the lab's clock. However, in hopping to the frame of one of the particles, the situation is quite different. Pick the direction of motion to be the z-axis.

   One particle moves along the $+z$ direction. The +z particle's frame is, by convention, the prime frame. If we hop to the prime frame, the relativistic velocity transformation along the z-axis (with $c = 1$) is

   $$u'_z = \frac{u_z - v}{1 - u_z v} \tag{1}$$

   where v is the velocity frame we are hopping to and u_z is the velocity of the other object in the unprime frame. In our case, using $v = 0.8$ and $u_z = -0.8$ gives $u'_z = -0.976$ (negative meaning that it is moving in the $-z' = -z$ direction). If we hopped to the $-z$ moving particle, we would, by convention be going to the unprime frame and would use

   $$u_z = \frac{u'_z + v}{1 + u'_z v} \tag{2}$$

   with $u'_z = +0.8$. This gives $u_z = +0.976$. The main point of this problem is to remind you that relative velocities don't generally add in an intuitive way. As a rule of thumb, when $v < 0.1c$, to a good approximation, we are able to add velocities.

2. It is helpful to recall the identity (with $u \leq 1$)

   $$y_u = \tanh^{-1}(u) = \frac{1}{2}\ln(\frac{1+u}{1-u}). \tag{3}$$

   In one dimension, if the rapidities really add like Galilean velocities, then we would expect $y_{u'_z} = y_{u_z} - y_v$. With Eq.(3) we have $y_{u_z} = \frac{1}{2}\ln(\frac{1+u_z}{1-u_z})$, $y_{u'_z} = \frac{1}{2}\ln(\frac{1+u'_z}{1-u'_z})$, and $y_v = \frac{1}{2}\ln(\frac{1+v}{1-v})$. Plugging the expression for u'_z from Eq. (2) into $y_{u'_z}$ we get $\frac{1}{2}\ln(\frac{1+u_z-v-u_z v}{1-u_z+v-u_z v})$. This is exactly equal to $y_{u_z} - y_v = \frac{1}{2}\ln(\frac{1+u_z}{1-u_z}) - \frac{1}{2}\ln(\frac{1+v}{1-v})$ (use the log identity $\log(A) - \log(B) = \log(\frac{A}{B})$). The main point of this problem is to introduce rapidity. The main utility of rapidity, as mentioned, is that relative rapidities do generally add intuitively (besides, one gets tired of writing 9's after the decimal place when expressing velocities close to the speed of light).

3. Recall $\vec{p} = \gamma m\vec{u}$ so the z component is $p_z = \gamma m u_z$. Using $E = \gamma m$ it follows that $u_z = p_z/E$. The point of this problem is to introduce a common expression for the longitudinal velocity. Using Eq.(3) and substituting $u = u_z = p_z/E$ we get, after factoring out an E in both the numerator and denominator, $y = \frac{1}{2}\ln(\frac{E+p_z}{E-p_z})$. This is a common form for the rapidity because most people feel more comfortable with logs rather than inverse hyperbolic functions.

4. Recall $\vec{p} = \gamma m\vec{u}$ so $\vec{p_T} = \gamma m(u_x\hat{x} + u_y\hat{y})$. Therefore, $p_T^2 = \gamma^2 m^2(u_x^2 + y_y^2)$. Now the transverse mass can be written $m_T = \sqrt{m^2 + p_T^2} = m\sqrt{1+\gamma^2(u_x^2+u_y^2)} = \gamma m\sqrt{\frac{1}{\gamma^2}+u_x^2+u_y^2}$. Also, $u_z = \tanh(y)$ so $\frac{1}{\sqrt{1-u_z^2}} = \cosh(y)$. Using $\gamma^2 = \frac{1}{1-u^2} = \frac{1}{1-u_x^2-u_y^2-u_z^2}$ it follows that $u_x^2+u_y^2+\frac{1}{\gamma^2} = (1-u_z^2)$ or $m_T = \frac{\gamma m}{\cosh(y)}$. Therefore, $m_T\cosh(y) = \gamma m$. Whew! This problem is trying to convince you that the common expression for the energy often used in introductory modern physics courses, $E = \gamma m$, is the same as $m_T\cosh y$, a common form for researchers doing relativistic collisions. This is really just a change of variable to a new set of kinematic coordinates (from v and m to y and m_T). This is often more convenient for many of the kinds of problems addressed in heavy ion and high energy physics. The main thing to remember is that γ in the expression for E is defined in terms of the particles speed in whatever direction it is moving. However, the expression $m_T\cosh y$ clearly delimits between the transverse and longitudinal contributions to the energy.

5. Recall $\vec{p} = \gamma m\vec{u} = \gamma m(u_x\hat{x} + u_y\hat{y} + u_z\hat{z})$ so $p_z = \gamma m u_z$. The z-component of the velocity is $u_z = \tanh(y)$ and from the solution to problem 4, $\gamma m = m_T\cosh(y)$. So, $p_z = m_T\cosh(y)\tanh(y) = m_T\sinh(y)$. See the note above from problem 4.

6. This problem is a little tricky for a couple reasons. First, it twists one intuition around. A common response to this problem is often "The transverse momentum is a transverse vector. Why would we even

expect it to change under longitudinal boosts?" This logic is flawed. For example, the transverse velocity, also a transverse vector, *does* in fact change under longitudinal boosts (as the problem indicates). Second, once willing to forge ahead, setting up the equations can be tricky.

For a particle moving in three dimensions, $\vec{p} = \gamma_u m\vec{u}$ so the transverse (x and y) component is $p_T = \gamma_u m u_T$ where $u_T = \sqrt{u_x^2 + u_y^2}$ and $\gamma_u = \frac{1}{\sqrt{1-u^2}}$. If we hop to a prime frame moving longitudinally with velocity v, the z-component of the velocity transforms like Eq.(2) while the transverse components transform as

$$u'_T = \frac{u_T}{\gamma_v(1 - u_z v)} \tag{4}$$

where $\gamma_v = \frac{1}{\sqrt{1-v^2}}$. In this prime frame, the transverse momentum is given by $p'_T = \gamma_{u'} m u'_T$ where $\gamma_{u'} = \frac{1}{\sqrt{1-u'^2}}$ and $u' = (u'_T)^2 + (u'_z)^2$. Notice the gymnastics with the gamma's. If we substitute the expressions for u'_z and u'_T into the equation for $\gamma_{u'}$, after a fair bit of algebra, you should get $\gamma_{u'} = \gamma_u\gamma_v(1 - u_z v)$. With this expression for $\gamma_{u'}$ as well as Eq.(4) we get $p'_T = (\gamma_u\gamma_v(1 - u_z v))m(\frac{u_T}{\gamma_v(1-u_z v)}) = \gamma_u m u_T = p_T$. Whew!

7. If you boost to the same longitudinal frame as a particle (that is, match its longitudinal velocity), it will be moving transversely away from you. This frame is called the longitudinally comoving frame (LCMF). The transverse mass is simply the total energy in that frame ($y = 0$ so $E = m_T = \sqrt{m^2 + p_T^2}$). The only energy a particle has in the LCMF is its rest energy (m) and its energy due to the transverse motion. In the absence of dynamical transverse boosts (e.g. flow), and if we only allow longitudinal frame changes, the transverse mass becomes an invariant quantity. That is, it acts as a "mass-like quantity" in the LCMF (see problem 6 to convince yourself that p_T doesn't change under longitudinal boosts). This has some utility. Also note that in the LCMF, the kinetic energy, generally $E - m$, becomes $m_T - m$.