10.2 a)

$$\vec{\varepsilon}_0 = \frac{1}{\sqrt{1+r^2}}\left(\vec{\varepsilon}_+ + r e^{i\alpha}\vec{\varepsilon}_-\right)$$

but $\vec{\varepsilon}_\pm = \frac{1}{\sqrt{2}}\left(\vec{\varepsilon}_1 \pm i\vec{\varepsilon}_2\right)$

so $$\vec{\varepsilon}_0 = \frac{1}{\sqrt{2}\sqrt{1+r^2}}\left(\vec{\varepsilon}_1 + i\vec{\varepsilon}_2 + r e^{i\alpha}\left(\vec{\varepsilon}_1 - i\vec{\varepsilon}_2\right)\right)$$

$$= \frac{1}{\sqrt{2(1+r^2)}}\left(\vec{\varepsilon}_1\left(1 + re^{i\alpha}\right) + \vec{\varepsilon}_2 i\left(1 - re^{i\alpha}\right)\right)$$

We will use the result of problem 10.1 b)

$$\frac{d\sigma}{d\Omega}\left(\vec{\varepsilon}_0, \vec{n}_0, \vec{n}\right) = k^4 a^6\left[\frac{5}{4} - \left|\vec{\varepsilon}_0\cdot\vec{n}\right|^2 - \frac{1}{4}\left|\hat{n}\cdot\left(\vec{n}_0\times\vec{\varepsilon}_0\right)\right|^2 - \vec{n}_0\cdot\vec{n}\right]$$

we can resolve $\vec{n}$ into components along $\vec{n}_0$, $\vec{\varepsilon}_1$, $\vec{\varepsilon}_2$ directions, treating $\vec{n}_0$ as $\vec{z}$, $\vec{\varepsilon}_1 \to \vec{x}$, $\vec{\varepsilon}_2 \to \vec{y}$

$$\hat{n} = \vec{n}_0\cos\theta + \vec{\varepsilon}_1\sin\theta\cos\phi + \vec{\varepsilon}_2\sin\theta\sin\phi$$

so $$\vec{\varepsilon}_0\cdot\vec{n} = \frac{1+re^{i\alpha}}{\sqrt{2(1+r^2)}}\sin\theta\cos\phi + i\frac{1-re^{i\alpha}}{\sqrt{2(1+r^2)}}\sin\theta\sin\phi$$

10.2 a) cont.

$$\vec{n}_0 \cdot \vec{n} = \cos\theta$$

$$\vec{n}_0 \times \vec{\varepsilon}_0 = \vec{\varepsilon}_1\left(-i(1-re^{i\alpha})\right) - \vec{\varepsilon}_2\left(-(1+re^{i\alpha})\right) \Big/ \sqrt{2(1+r^2)}$$

$$= \left(-i(1-re^{i\alpha})\vec{\varepsilon}_1 + (1+re^{i\alpha})\vec{\varepsilon}_2\right) \Big/ \sqrt{2(1+r^2)}$$

so $\vec{n}\cdot(\vec{n}_0 \times \vec{\varepsilon}_0) = \left(-i(1-re^{i\alpha})\sin\theta\cos\phi + (1+re^{i\alpha})\sin\theta\sin\phi\right) \Big/ \sqrt{2(1+r^2)}$

$$\left[2(1+r^2)\right] \cdot \left|\vec{n}\cdot(\vec{n}_0 \times \vec{\varepsilon}_0)\right|^2 = \left((1+re^{i\alpha})\sin\theta\sin\phi - i(1-re^{i\alpha})\sin\theta\cos\phi\right)\left((1+re^{-i\alpha})\sin\theta\sin\phi + i(1-re^{-i\alpha})\sin\theta\cos\phi\right)$$

$$= (1+2r\cos\alpha+r^2)\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi(1-2r\cos\alpha+r^2) - 4r\sin^2\theta\sin\phi\cos\phi\sin\alpha$$

$$= (1+r^2)\sin^2\theta + 2r\cos\alpha\sin^2\theta(\sin^2\phi - \cos^2\phi) - 4r\sin^2\theta\sin\phi\cos\phi\sin\alpha$$

and $|\vec{\varepsilon}_0 \cdot \vec{n}|^2 = \left(\frac{1+re^{i\alpha}}{\sqrt{2(1+r^2)}}\sin\theta\cos\phi + i\frac{1-re^{i\alpha}}{\sqrt{2(1+r^2)}}\sin\theta\sin\phi\right)\left(\frac{1+re^{-i\alpha}}{\sqrt{2(1+r^2)}}\sin\theta\cos\phi - i\frac{1-re^{-i\alpha}}{\sqrt{2(1+r^2)}}\sin\theta\sin\phi\right)$

$$= \frac{1+2r\cos\alpha+r^2}{2(1+r^2)}\sin^2\theta\cos^2\phi + \frac{4r\sin\alpha\sin^2\theta\sin\phi\cos\phi}{2(1+r^2)} + \frac{1-2r\cos\alpha+r^2}{2(1+r^2)}\sin^2\theta\sin^2\phi$$

$$= \frac{\sin^2\theta}{2(1+r^2)}\left(1+r^2 + 2r\cos\alpha(\cos^2\phi - \sin^2\phi) + 4r\sin\alpha\sin\phi\cos\phi\right)$$

10.2 a) cont

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\frac{5}{4} - \frac{\sin^2\theta}{2(1+r^2)} \left(1+r^2 + 2r\cos\alpha(\cos^2\phi - \sin^2\phi) + 4r\sin\alpha\sin\phi\cos\phi - \frac{1}{4}\left((1+r^2) + 2r\cos\alpha(\sin^2\phi - \cos^2\phi) + 4r\sin\phi\cos\phi\sin\alpha \right) \right) - \cos\theta \right]$$

but $-\sin^2\phi + \cos^2\phi = \cos 2\phi$

$2\cos\phi\sin\phi = \sin 2\phi$ so

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left[\frac{5}{4} - \frac{\sin^2\theta}{2(1+r^2)} \left(1+r^2 + 2r\sin\alpha\sin 2\phi + 2r\cos\alpha\cos 2\phi - \frac{1}{4}(1+r^2) - \frac{1}{2}\cos\alpha\cos 2\phi - \frac{1}{2}\sin\alpha\sin 2\phi \right) - \cos\theta \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \frac{\sin^2\theta}{2(1+r^2)} \left(\frac{3}{4}(1+r^2) + \frac{3r}{2}\sin\alpha\sin 2\phi + \frac{3r}{2}\cos\alpha\cos 2\phi \right) - \cos\theta \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \frac{\sin^2\theta}{2(1+r^2)} \left(\frac{3}{4}(1+r^2) + \frac{3r}{2}\cos(\alpha - 2\phi) \right) - \cos\theta \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \frac{1-\cos^2\theta}{2(1+r^2)} \left(\frac{3}{4}(1+r^2) + \frac{3r}{2}\cos(\alpha - 2\phi) \right) - \cos\theta \right]$$

$$= k^4 a^6 \left[\frac{5}{4} - \frac{5}{8} + \frac{5}{8}\cos^2\theta + \frac{3}{4}\frac{r}{1+r^2}\sin^2\theta\cos(\alpha - 2\phi) - \cos\theta \right]$$

$$\boxed{= k^4 a^6 \left[\frac{5}{8}(1+\cos^2\theta) + \frac{3}{4}\frac{r}{1+r^2}\sin^2\theta\cos(\alpha - 2\phi) - \cos\theta \right]}$$

Since we used the result of 10.1, this is consistant with 10.1

10.3

Because I'm low on time, I'll give an outline of the solution to this problem, but won't do the algebra.

20/50

a) $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

because of the infinite conductivity, all induced currents will be surface currents, so $\vec{J} = 0$

in the (extremly) long wavelength limit, frequencys are very low so $\frac{\partial \vec{D}}{\partial t}$ can be taken as zero to first order.

thus $\nabla \times H \approx 0$ and use of the magnetic scalar potential is justified

From here it is a statics / Boundary value problem

solve $\vec{\nabla}^2 \Phi_M = 0$ matching the appropriate boundary conditions on $\vec{H}$ and $\vec{B}$, solve, then

$$-\vec{\nabla} \cdot \Phi_M = \vec{H}$$

b) use the relations $\delta = \left(\frac{2}{\mu_c \omega \sigma}\right)^{1/2}$ (skin depth)

and $\dfrac{dP_{loss}}{da} = \dfrac{1}{2\sigma\delta} \left|\vec{K}_{eff}\right|^2$

10.3 cont.

$\vec{K}_{eff}$ can be found from the boundary condition on $\vec{H}_1$

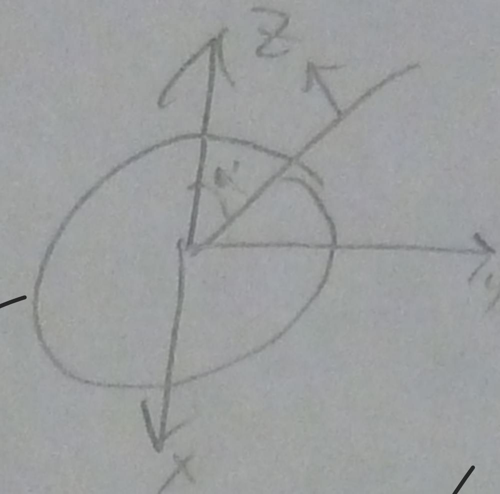
By dividing $\dfrac{dP_{loss}}{da}$ by the Pointyng Vector,

we can relate it to cross section.

10.12

let $\vec{E}_{inc} = \hat{\varepsilon} E_0 e^{i\vec{k}_0\cdot\vec{r}}$

$\hat{\varepsilon} = \hat{y}$



$\hat{z} = \hat{n}$

x-y plane is plane of aperture

yz plane is plane of incidence

eqn 10.101 $\quad E_{diff} = \frac{1}{2\pi} \vec{\nabla} \times \int_{ap} (\hat{n} \times \vec{E}_{inc}) \frac{e^{ikR}}{R}\, da'$

$$\hat{n} \times \vec{E}_{inc} = E_0 e^{i\vec{k}_0\cdot\vec{r}} (\hat{z} \times \hat{y})$$

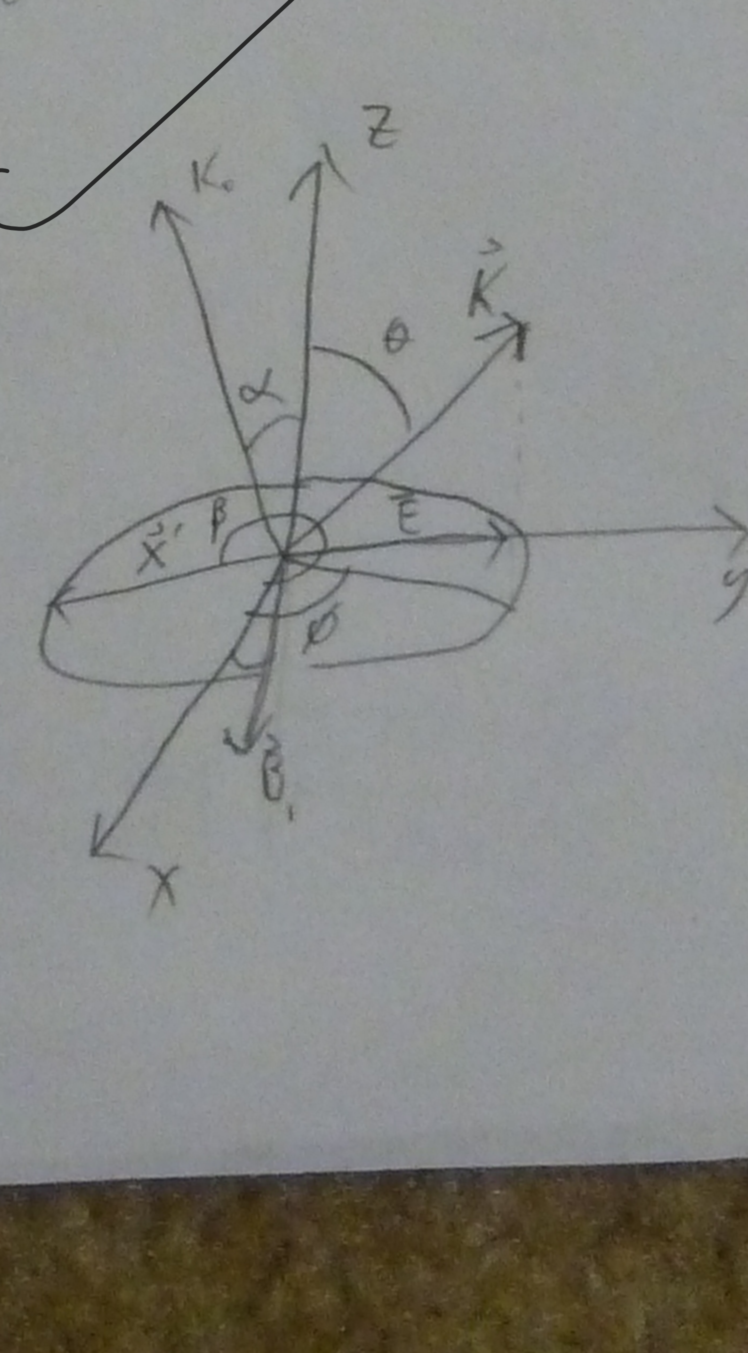
$$= -E_0 e^{i\vec{k}_0\cdot\vec{r}} \hat{x}$$

In the small aperture limit, 10.101 reduces to the simpler form 10.109

$$\vec{E}(\vec{x}) = \frac{i e^{ikr}}{2\pi r} \vec{k} \times \int_{aperture} \hat{n} \times \vec{E}_{inc}(\vec{x}') e^{-i\vec{k}\cdot\vec{x}'}\, da'$$

$$= -\frac{i e^{ikr}}{2\pi r} \vec{k} \times E_0 \hat{x} \int e^{i\vec{k}_0\cdot\vec{x}'} e^{-i\vec{k}\cdot\vec{x}'}\, da'$$

$$= -\frac{i e^{ikr} E_0}{2\pi r} \vec{k} \times \hat{x} \int e^{i(\vec{k}_0 - \vec{k})\cdot\vec{x}'}\, da'$$



note that $\vec{x}'$ lies in the x-y plane, so in taking the dot product, only the x-y components of $\vec{k}$ and $\vec{k}_0$ will remain

10.12 cont.

So the integrand becomes

$$\int e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{x}'} da'$$

with $\vec{k}_1 = \hat{x}\vec{k}_{0x} + \vec{k}_{0y}\hat{y}$

$\vec{k}_2 = \hat{x}k_x + k_y\hat{y}$

note $|\vec{k}_1| = k\sin\alpha$ with $k=|\vec{k}_0|=|\vec{k}|$

$|\vec{k}_2| = k\sin\theta$,

so $|\vec{k}_1-\vec{k}_2| = k(\sin^2\alpha+\sin^2\theta-2\sin\alpha\sin\theta\cos\phi)$

and $(\vec{k}_1-\vec{k}_2)\cdot\vec{x}' = |\vec{k}_1-\vec{k}_2||\vec{x}|\cos(\beta')$

where $\beta' \equiv$ the angle between $\vec{x}'$ and $\vec{k}_1-\vec{k}_2$

$|\vec{x}'| = r$, so

$$\int e^{ikr\underbrace{(\sin^2\alpha+\sin^2\theta-2\sin\alpha\sin\theta\cos\phi)}_{\mathcal{L}}\cos\beta'} r\,dr\,d\beta'$$

3

$$\int_0^a\int_0^{2\pi} e^{ikr\mathcal{L}\cos\beta} r\,dr\,d\beta'$$

(note: Jackson gets a minus sign in the exponential. No idea where it comes from)

Jackson notes that this integral (with the missing minus) becomes

$$2\pi J_0(kr\mathcal{L})$$

Open your math method book and look at the bessel function

The remaining radial integral results in

$$2\pi a^2 \frac{J_1(ka\mathcal{L})}{ka\mathcal{L}}$$

So we get $$E(\vec{x}) = -\frac{ie^{ikr}E_0 a_2}{r}(\vec{k}\times\hat{x})\frac{J_1(ka\mathcal{L})}{ka\mathcal{L}}$$

10.12 b)

comparing to 10.113, the main difference is the factor of $\cos\alpha$ resulting from the fact that in the book case the incident electric field tangential to the plane of incedence is perturbed. There is, of course, also a different resultant polarization.

good !

comparing to the scalar case (~10.119) the main difference is that instead of $\vec{k}\times\hat{x}$ we set $k\left(\frac{\cos\alpha+\cos\theta}{2}\right)$

and $\cos\theta \to 0$ as $\vec{k}\to\hat{x}$, so part of it even has the right dependance!

its sort of like an average of the perpendicular and parallel cases!

-2

Does the result depend on the azimuthal angle? Explain