Matthew Lawson

6.4 a)

Ohm's law is $\vec{J} = \sigma(\vec{E} + \vec{V} \times \vec{B})$

hopefully not a superconductor so $\sigma \neq \infty$.

4/5

we have no current, so

$$\sigma(\vec{E} + \vec{V} \times B) = 0$$

$$\vec{E} = -\vec{V} \times \vec{B}$$

the magnetic field inside a uniformly magnetized sphere is given by eqn 5.105

$$\vec{B} = \frac{2\mu}{3} \vec{M}$$

where for this case, $\vec{M} = \frac{3}{4\pi R^3} \vec{m} = M\hat{z}$

going into cylindrical coordinates,

$\vec{V} = \hat{\phi}\, \omega r$, so

$$\vec{E}_{in} = -\hat{r}\, \omega r \frac{2\mu}{3} M = -\frac{2\mu\, \omega r M}{3} \hat{r}$$

but $\vec{\nabla} \cdot \vec{E}_{in} = \frac{\rho(\vec{x})}{\varepsilon}$

6.4 a) cont.

$$\vec{\nabla} \cdot \vec{E}_{in} = \frac{1}{r} \frac{\partial}{\partial r}(r \vec{E}_{in})$$

$$= -\frac{2\mu \omega M}{3r} \frac{\partial}{\partial r}(r^2)$$

$$= -\frac{4\mu \omega M}{3} = \rho/\varepsilon$$

$$\rho = -\frac{4\mu \varepsilon \omega M}{3} = -\frac{4\omega M}{3c^2}$$

$$\vec{M} = \frac{3M}{4\pi R^3} \quad \text{so}$$

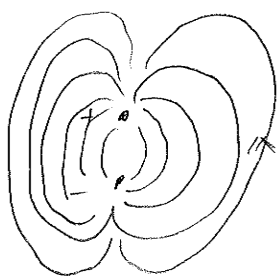
$$\boxed{\rho = -\frac{M\omega}{2\pi c^2 R^3}}$$

6.4 b.

because there is no net charge, there can be no monopole term.

a dipole field is not invariant under 180° degree rotations, while this field is. (180° about, say the x axis)
A quadropole field is however invariant under 180° flips, so it is allowed.

Show mathematically



6.4 b.

note that $\vec{E}_{in} = -\frac{2\mu\omega r M}{3}\hat{r}$ in cylindrical coords

in spherical coords this becomes

$$E_{in} = -\frac{2\mu\omega M r \sin^2\theta}{3}\hat{r} - \frac{2\mu\omega M r \sin\theta\cos\theta}{3}\hat{\theta}$$

note that the electric field must be continuous at the boundary, or in this case, the tangential component must be continuous.

so $E_{\perp} = -\frac{2\mu M\omega r \sin\theta\cos\theta}{3}\hat{\theta}$

by 4.11, and noting that $\frac{\partial}{\partial\theta} Y_{20}(\theta,\phi)$

$= -\frac{3}{2}\sqrt{\frac{5}{\pi}} \sin\theta\cos\theta$,

How?

we know that the θ component of $E_{outside}$ will contain only the Y_{20} component.

in other words, we can write

$$\vec{E}_{\theta} = \sum_{lm} -\frac{1}{(2l+1)\varepsilon_0} q_{lm} \frac{1}{r^{l+2}} \frac{\partial}{\partial\theta} Y_{lm}(\theta,\phi)$$

6.4 b) cont.

by orthogonality, only the $l=2, m=0$ term is nonzero,

$$\text{so} \quad + \frac{1}{4\pi \varepsilon_0} q_{20} \frac{1}{R^4} \frac{3}{2}\sqrt{\frac{5}{\pi}} \sin\theta\cos\theta$$

$$= -\frac{2\mu M \omega R \sin\theta\cos\theta}{3}$$

$$q_{20} = -\frac{20\varepsilon_0 \mu M \omega R^5 \sqrt{\pi}}{9\sqrt{5}}$$

$$\boxed{= -\frac{10\varepsilon_0 \mu M \omega R^5 \sqrt{4\pi}}{9\sqrt{5}}}$$

NO!!

$$\mu = \frac{3M}{4\pi R^3}, \text{ so } -\frac{10\varepsilon_0 \mu M \omega R^2}{3\sqrt{4\pi}\sqrt{5}} = q_{20}$$

NO!

$$\boxed{q_{20} = -\frac{10 M \omega R^2}{3c^2\sqrt{4\pi}\sqrt{5}} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{33}}$$

$$-\frac{4 M \omega R^2}{3c^2} = Q_{33}$$

by noting that $q_{22} = 0$ and $q_{22} \propto (Q_{11} - Q_{22} - \cancel{2iQ_{12}}^{\,0})$

we know $Q_{11} = Q_{22}$

since Q_{ij} is traceless, this means $Q_{11} = Q_{22} = -Q_{33}/2$

c)

we know that the radial electric field at $r=R$ is

$$E_{in_r} = -\frac{2\mu\omega M R \sin^2\theta}{3} = -\frac{\mu\omega M\sin^2\theta}{2\pi R^2}$$

outside, since we know only q_{20} survives,

$$E_{out_r} = +\frac{3}{5\varepsilon_0} q_{20} \frac{Y_{20}}{R^4}$$

using the correct (I hope!) $q_{20} = -\frac{10 M\omega R^2}{3c^2\sqrt{5}\sqrt{4\pi}}$, we get

$$-\frac{3}{5\varepsilon_0} \frac{10 M\omega R^2}{3c^2\sqrt{5}} \frac{\sqrt{5}}{R^4 4\pi}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$= -\frac{2M\omega\mu}{R^2 4\pi}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

the difference gives the surface charge density$/\varepsilon$

$$\frac{\sigma}{\varepsilon_0} = \frac{2\mu\omega M\sin^2\theta}{4\pi R^2} - \frac{2\mu\omega\mu}{4\pi R^2}\left(3\cos^2\theta - 1\right)$$

$$\sigma = \frac{2M\omega}{4\pi R^2 c^2}\left(\sin^2\theta - \frac{3}{2}\cos^2\theta + \frac{1}{2}\right)$$

6.4 c)

$$\sin^2\theta - \frac{3}{2}\cos^2\theta + \frac{1}{2}$$

$$= 1 - \cos^2\theta - \frac{3}{2}\cos^2\theta + \frac{1}{2}$$

$$= \frac{3}{2} - \frac{5}{2}\cos^2\theta$$

note that

$$1 - \frac{5}{2}P_2 = 1 - \left(\frac{3}{2}\cos^2\theta + \frac{1}{2}\right)\frac{5}{2}$$

$$= 1 - \frac{15}{4}\cos^2\theta + \frac{5}{4}$$

$$= \frac{9}{4} - \frac{15}{4}\cos^2\theta$$

$$= \frac{3}{2}\left(\frac{3}{2} - \frac{5}{2}\cos^2\theta\right)$$

so $\frac{2}{3}\left(\sin^2\theta - \frac{3}{2}\cos^2\theta + \frac{1}{2}\right) = 1 - \frac{5}{2}P_2$

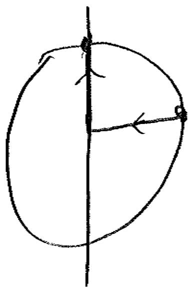
so

$$\sigma = \frac{\cancel{7}MW}{\cancel{4\pi r_2^2}}\left(\frac{2}{\cancel{3}}\right)\left(1 - \frac{5}{2}P_2(\cos^2\theta)\right)$$

what happened to r_2^2?

6.4 d)

consider the following path:



$$E_{in} = -\frac{\mu \omega M r \sin^2\theta}{2\pi R^3}$$

1st leg $$\int_R^0 dr\, E_{in} = \frac{\mu \omega M \sin^2\theta}{4\pi R} = \frac{\mu \omega M}{4\pi R}$$

2nd leg $$\int_0^R dr\, E_{in}(r, \theta=0) = 0$$

so the integral is $\frac{\mu \omega M}{4\pi R}$

6.8

according to eqn 4.58, a dielectric sphere in a uniform electric field has a polarization given by

$$\vec{P} = (\varepsilon - \varepsilon_0)\vec{E} = 3\varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right)\vec{E}_0$$

which induces a surface charge of polarization $\sigma_{pol} = (\vec{P}\cdot\vec{r})/r$

however if there is a surface charge on a spinning sphere, there must be a bound surface current too.

The velocity of the sphere at a point on its surface with angle θ is

$$v = \omega R \sin\theta\, \hat{\phi}$$

so the surface current density is

$$\vec{K} = \omega R \sin\theta\, \sigma_{pol}\, \hat{\phi}$$

$$\sigma_{pol} = \vec{P}\cdot\hat{n} = 3\varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right) E_0 \sin\theta$$

so

$$\vec{K} = \omega R\, 3\varepsilon_0 E_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2}\right) \sin^2\theta\, \hat{\phi}$$

6.8 cont.

; change coordinates to cylindrical,

$\sin\theta\, \hat{\phi} = \hat{z} \times \hat{r}$

$\underbrace{\sin\theta\, \hat{\phi}}_{\text{spherical}} = \underbrace{\hat{z} \times \hat{r}}_{\text{cylindrical}}$

so $\hat{K} = w R\, \sigma_{pol}\, \hat{z} \times \hat{r}$

now $\vec{K}_{bound} = \vec{M} \times \hat{n}$ (Griffiths 6.14)

so $\vec{M} = w r\, \sigma_{pol}\, \hat{z}$ (note $R \to r$ as we are going into the bulk)

$= w\, \vec{p} \cdot \hat{n}\, r\, \hat{z}$

$= w\, \hat{p} \cdot \vec{r}\, \hat{z}$

$= w\, p \times \hat{z}$ ← being a little sloppy w/ coordinates here, but we are now in cartesian!

$= \underbrace{3 w \varepsilon_0 \left(\frac{\varepsilon/\varepsilon_0 - 1}{\varepsilon/\varepsilon_0 + 2} \right)}_{\gamma} E_0 \times \hat{z}$

$= \gamma \times \hat{z}$: note that $\hat{z} \cdot \hat{r} = \cos\theta$

so $\vec{M} \cdot \hat{n} = \gamma\, r \sin\theta \cos\phi \cos\theta$ (in spherical)

6.8 cont.

$\vec{\nabla}\cdot\vec{M} = 0$, so

$$\Phi_M(\vec{x}) = \frac{1}{4\pi}\oint_S \frac{\hat{n}\cdot\vec{M}(\vec{x}')\,da}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\gamma}{4\pi}\oint_S \frac{r\sin\theta\cos\theta\cos\phi}{|\vec{x}-\vec{x}'|}\,da$$

note that $\sin\theta\cos\theta\cos\phi = -\frac{1}{2}\sqrt{\frac{8\pi}{15}}\left(Y_{21} - Y_{2,-1}\right)$

so $$\Phi_M(\vec{x}) = -\frac{\gamma\sqrt{8\pi}R^3}{8\pi\sqrt{15}}\int_0^{2\pi}\int_0^{\pi}\frac{Y_{21} - Y_{2-1}}{|\vec{x}-\vec{x}'|}\,d\Omega'$$

now $$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi\sum_{lm}\frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

by orthogonality, only the $l=2$ $m=\pm 1$ terms of the sum survive

so $$\Phi_M(\vec{x}) = +\frac{\gamma\sqrt{8\pi}R^3}{10\sqrt{15}}\frac{r_<^2}{r_>^3}\left(Y_{2-1}(\theta,\phi) - Y_{21}(\theta,\phi)\right)$$

6.8 cont.

$$Y_{2-1} - Y_{21} = 2\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta\cos\phi$$

so $$\Phi_M = \frac{\gamma R^3}{5} \frac{r_<^2}{r_>^3} \sin\theta\cos\theta\cos\phi$$

now if $r > R$,

$$r_> = r, \quad r_< = R$$

so $$\Phi_M = \frac{\gamma R^5}{5 r^3} r^2 \sin\theta\cos\theta\cos\phi$$

$$= \frac{\gamma R^5}{5 r^5} xz$$

if $r < R$, $\quad r_> = R \quad r_< = r$

$$\Phi_M = \frac{\gamma R^3}{5} \frac{r^2}{R^3} \sin\theta\cos\theta\cos\phi = \frac{\gamma}{5} xz$$

so $$\Phi_M = \frac{\gamma}{5}\left(\frac{R}{r_>}\right)^5 xz = \frac{3}{5}\omega\epsilon_0\left(\frac{\epsilon/\epsilon_0 - 1}{\epsilon/\epsilon_0 + 2}\right) E_0\, xz$$

6.9

43/50

Conservation of Momentum

The statement of conservation of momentum is

$$\frac{d}{dt}\left(\vec{P}_{mech}+\vec{P}_{field}\right)_\alpha = \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta \, da$$

That is, the change in momentum is the surface integral of the stress tensor, which represents the flow of momentum across the boundary.

Jackson declines to consider the generalization of this statement to media, and instead refers the reader variously to Landau, de Groot, and Stratton. I do not have access to these texts, and so I won't be giving quantitative statements, however (as Jackson notes) it becomes important to consider the temperature, pressure, and density dependance of ϵ and μ in your media, as dissipative heating can then change the dialectric properties of the medium.

6.9

49/50

Conservation of Energy

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}$$ (continuity eqn)

This is a statement of conservation of energy. (in vaccum)

The change in energy density with time, plus the amount of energy coming through the boundaries $(\vec{\nabla} \cdot \vec{S})$ is equal to the work done by the electric field on the charges $(\vec{J} \cdot \vec{E})$ in principle we could have a $(\vec{J}_{Monopole} \cdot \vec{B})$ term as well.

in A linear media, we need to consider dispersive and dissipative effects as well. Then we get

$$\frac{\partial u_{eff}}{\partial t} + \vec{\nabla} \cdot \vec{S} = \underbrace{-\vec{J} \cdot \vec{E}}_{\text{Ohmic lossis}} \underbrace{- \omega_0 \, I_m \, \varepsilon(\omega_0) \langle \vec{E}(\vec{x},t) \cdot \vec{E}(\vec{x},t) \rangle - \omega_0 \, I_m \, \mu_0(\omega_0) \langle \vec{H}_1(\vec{x},t) \cdot \vec{H}_1(\vec{x},t) \rangle}_{\text{absorbtive-type effects}}$$

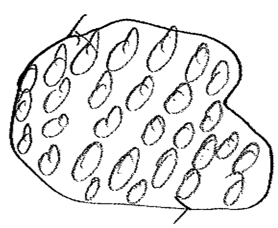
6.9 cont,

Derivations:

I will note that this problem as stated does not make much sense to me. The derivation for an expression for u in media is done explicitly by Jackson, and the other quantities are largely defenitional.

$u_{magnetic}$

Consider the magnetic field due to a current loop. Any current loop can be written as a network of infintesimal loops:



as all the inner parts of the loops cancel.

We can write the energy change due to the addition of a single current loop

$$\Delta(\delta W) = J \Delta\sigma \int_S \hat{n} \cdot \delta\vec{B}\, da$$

6.9 cont.

$$\Delta(\delta w) = J \Delta \sigma \int_S (\vec{\nabla} \times \delta \vec{A}) \cdot \vec{n}\, da \qquad (\vec{\nabla} \times \vec{A} = \vec{B})$$

NO!!

stokes theorem gives

$$\Delta(\delta w) = J \Delta\sigma \oint_S \delta\vec{A} \cdot d\vec{l}$$

6

$$(\vec{J} = J \Delta\sigma\, d\vec{l})$$

$$\delta w = \int \delta\vec{A} \cdot \vec{J} d^3x$$

should be curl

Ampere's law gives

$$\delta w = \int \delta\vec{A} \cdot (\vec{\nabla} \times \vec{H}) d^3x$$

vector identities give

$$\delta w = \int \left[\vec{H} \cdot (\vec{\nabla} \times \delta\vec{A}) + \vec{\nabla} \cdot \underbrace{(\vec{H} \times \delta\vec{A})}_{0}\right] d^3x$$

$$= \int \vec{H} \cdot \delta\vec{B}\, d^3x$$

if a linear relationship between $\vec{B}$ and $\vec{H}$ exists,

$$\vec{H} \cdot \delta\vec{B} = \frac{1}{2}\delta(\vec{H} \cdot \vec{B})$$

so

$$W = \frac{1}{2} \int \vec{H} \cdot \vec{B} d^3x$$

if μ is a function of position however, this relation does not hold! then consider $\delta\vec{B} = \delta(\vec{H}\mu(x))$ and go from there.

6.9 cont.

$U_{electric}$ follows in a pretty similar way,

ending with the equation

$$W = \int d^3x \int_0^D \vec{E} \cdot \delta \vec{D}$$

if the medium is linear, then

$$\vec{E} \cdot \delta \vec{D} = \frac{1}{2} \delta (\vec{E} \cdot \vec{D})$$

giving

$$W = \frac{1}{2} \int \vec{E} \cdot \vec{D} \, d^3x$$

otherwise, consider $\delta \vec{D} = \delta (\varepsilon(\vec{x}) \vec{E})$

$$= \vec{E} \, \delta \varepsilon(\vec{x}) + \varepsilon(\vec{x}) \, \delta \vec{E}(x)$$

so
$$W = \int d^3x \int_0^D \left(E^2 \delta \varepsilon + \varepsilon \vec{D} \, \delta \vec{E} \right)$$

$$= \int d^3x \left(\frac{1}{2} \vec{E} \cdot \vec{D} + \underbrace{\int_0^D E^2 \delta \varepsilon}_{} \right)$$

not sure how to deal with this term

6.9 cont.

combining the expressions,

$$W = \frac{1}{2}\int d^3x \left(\varepsilon E^2 + \mu H^2 \right)$$ so $u = \frac{1}{2}\left(\varepsilon E^2 + \mu H^2\right)$

or in the case of variable μ and ε,

$$W = \int d^3x \left(\tfrac{1}{2}\varepsilon E^2 + \tfrac{1}{2}\mu H^2 + \int H^2 \delta\mu(\vec{x}) + \int E^2 \delta\varepsilon \right)$$

with the corresponding energy density, u

the Poynting vector arises from the desire to express the continuity eqn compactly.

The continuity eqn is

$$-\int_V \vec{J}\cdot E\, d^3x = \int_V \left[\frac{\partial u}{\partial t} + \vec{\nabla}\cdot \underbrace{(\vec{E}\times H)}_{\vec{S},\ \text{Poynting vector}} \right] d^3x$$

no modifications are needed here for variable μ, ε

6.9 cont.

now for g and T_{ij}. Following Jackson,

$$\frac{\partial P}{\partial t} = \int_V \left(\rho\vec{E} + \vec{J}\times\vec{B}\right) d^3x$$

now $\rho = \vec{\nabla}\cdot\vec{D}$, $\vec{J} = \vec{\nabla}\times\vec{H} - \frac{\partial D}{\partial t}$

if μ and ε are constants,

$$\rho = \varepsilon_0 \vec{\nabla}\cdot\vec{E}, \quad \vec{J} = \frac{1}{\mu}\vec{\nabla}\times B - \varepsilon\frac{\partial\vec{E}}{\partial t}$$

otherwise, we would now be in trouble, and have to take all those derivatives to try and combine terms! $\boxed{\text{anisotropic media is hard!}}$

NOT REALLY

if isotropic,

$$\rho\vec{E} + \vec{J}\times B = \varepsilon\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + \vec{B}\times\frac{\partial\vec{E}}{\partial t} - c^{*2}\vec{B}\times(\vec{\nabla}\times\vec{B})\right]$$

$$(c^*)^2 = \frac{1}{\mu\varepsilon}$$

note that $\vec{B}\times\frac{\partial\vec{E}}{\partial t} = -\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times\frac{\partial\vec{B}}{\partial t}$

and $c^2\vec{B}(\vec{\nabla}\cdot B) = 0$, so

$$\rho\vec{E} + \vec{J}\times\vec{B} = \varepsilon\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) + c^2\vec{B}(\vec{\nabla}\cdot B) - \vec{E}\times(\vec{\nabla}\times\vec{E}) - c^{*2}\vec{B}\times(\vec{\nabla}\times\vec{B})\right] - \varepsilon\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

6.9 cont.

so $\frac{\partial \vec{P}}{\partial t} + \frac{d}{dt}\int_V \varepsilon(\vec{E}\times\vec{B})\,d^3x = \varepsilon\int_V \Big[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + c^{*2}\vec{B}(\vec{\nabla}\cdot\vec{B})$
$\qquad - c^{*2}\vec{B}\times(\vec{\nabla}\times\vec{B})\Big]\,d^3x$

identify

$$\vec{P}_{field} = \varepsilon\int_V \vec{E}\times\vec{B}\,d^3x = \mu\varepsilon\int_V \vec{E}\times\vec{H}\,d^3x$$

so the momentum density is

$$\boxed{g = \frac{1}{\mu\varepsilon}(\vec{E}\times\vec{H})}$$

vector

if μ and ε depend on $\vec{x}$, none of those μ and ε through derivatives or integrals were legal! crap hits the fan!

now note that

$$\left[\vec{E}(\vec{\nabla}\cdot\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})\right]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta}\left(E_\alpha E_\beta - \frac{1}{2}\vec{E}\cdot\vec{E}\,\delta_{\alpha\beta}\right)$$

which suggests the identification

$$T_{\alpha\beta} = \varepsilon\left(E_\alpha E_\beta + c^{*2} B_\alpha B_\beta - \frac{1}{2}(\vec{E}\cdot\vec{E} + c^{*2}\vec{B}\cdot\vec{B})\,\delta_{\alpha\beta}\right)$$

which requires no modification.