Physics 200C, Hw 3

April 8, 2005

1. **Practice for E-M Field 4 Tensor and Lorentz Invariance**

If $\mathbf{K}$ is a unit three-vector in the direction of propagation of the wave, then according to classical electromagnetism, $\mathbf{K}\cdot\mathbf{E} = \mathbf{K}\cdot\mathbf{B} = 0$. Show that this statement is also invariant under Lorentz transformation by showing its equivalence to the manifestly Lorentz invariant statement $n^\mu F_{\mu\nu} = 0$, where n^μ is a four-vector oriented in the direction of propagation of the wave and $F_{\mu\nu}$ is the field strength tensor.

Solution:

Using the electromagnetic wave propagation four-vector $K^\mu = (K_1, K_2, K_3, \frac{\omega}{c})$, where $K = \frac{\omega}{c}$, we can express $n^\mu \equiv \frac{1}{K}K^\mu$. $n^\mu F_{\mu\nu} = 0$ which is equivalent to $K^\mu F_{\mu\nu} = 0$.

For $\nu = 1$, we have $K_2 cB_3 - K_3 cB_2 + \frac{\omega}{c}E_1 = 0$, or $E_1 = \frac{c}{K}(\mathbf{B}\times\mathbf{K})_1$.

Similar expressions are obtained for E_2 and E_3. Hence

$$\mathbf{E} = \frac{c}{K}(\mathbf{B}\times\mathbf{E}). \tag{1}$$

For $\nu = 4$, we have

$$\mathbf{K}\cdot\mathbf{E} = 0. \tag{2}$$

Since $n^\mu F_{\mu\nu} = 0$ is Lorentz covariant, it has the same form in all inertial frames.

This means that Eqs. (1) and (2) are valid in all inertial frames.

Now Eq. (1) gives

$$\mathbf{E}^2 = \frac{c^2}{K^2}(\mathbf{B}\times\mathbf{K})\cdot(\mathbf{B}\times\mathbf{K}) = c^2\mathbf{B}^2 - c^2(\mathbf{B}\cdot\mathbf{K})^2. \tag{3}$$

We know that if $\mathbf{E}^2 = c^2B^2$ is valid in an inertial frame it is valid in all inertial frames. Since this relation is given for one inertial frame, Eq. (3) means that the relation $\mathbf{K}\cdot\mathbf{B} = 0$ is satisfied in all inertial frames.