MULTIPLICITY DISTRIBUTIONS IN NUCLEUS-NUCLEUS COLLISIONS AT HIGH ENERGIES

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The average and the dispersion of multiplicity distributions in nucleus-nucleus collisions are calculated assuming that the inelastic collision of two nuclei is an incoherent composition of collisions of individual nucleons. The average multiplicity is assumed to be proportional to the number of “wounded nucleons” i.e. the nucleons which underwent at least one inelastic collision. For the sake of comparison the average and the dispersion of the number of collisions is also discussed. Our calculations indicate that in nucleus-nucleus collisions, the amplification of various characteristics of the nucleon-nucleon interaction is far greater than in hadron-nucleus collisions.

1. Introduction

In this paper we discuss multiplicity distributions of particles produced in collisions of two nuclei at high energies. Our basic assumption is that the inelastic collision of two nuclei can be described as an incoherent composition of the collisions of individual nucleons. This picture is an old one. It proved useful in the description of nucleon-nucleus collisions [1–5]. It seems therefore natural and interesting to extend it further to the nucleus-nucleus collisions in the hope of obtaining useful estimates of experimental characteristics such as multiplicities, dispersions and their dependences on nuclear parameters.

It should be stressed that in this approach the collective effects which may occur in nuclei are neglected. The existence of such effects in e.g. nucleon-nucleus interactions is controversial [6,7]. It is therefore interesting to extend the investigation of this problem to nucleus-nucleus interactions and our calculations should prove useful by giving predictions for the cases where collective effects are absent. Should future experiments prove that collective effects are indeed absent in inelastic collisions* our scheme can be used to analyze the properties of elementary collisions which are otherwise difficult to see. For example, it may be possible to separate different components in the elementary production processes which get differently amplified in the processes of multiple collisions.

* We neglect altogether the diffractive production processes.
In the case of nucleon-nucleus collisions a fundamental role is played by the number of collisions ($\nu$) of the incident nucleon with the nucleons in the target nucleus [1]. However, the generalization of this concept to nucleus-nucleus collisions is not unique.

In this paper we propose to describe the nucleus-nucleus collisions in terms of the number of "wounded" nucleons ($w$) i.e. the number of nucleons which underwent at least one inelastic collision in this process. For instance in the case of nucleon-nucleus collisions there are $\nu$ "wounded" nucleons in the target nucleus and one "wounded" incident nucleon. Consequently, in this case, there is a simple relation between $\nu$ and $w$

$$w = \nu + 1.$$  

Thus either of them can be used.

In the nucleus-nucleus collisions, however, there is no unique relation between $\nu$ and $w$. Therefore a choice has to be made and our conjecture is that, physically, the more relevant variable is $w$. The motivation for this choice comes from the interpretation of the available data on nucleon-nucleus interactions [8]. The average multiplicities in collisions of a high-energy nucleon with a target nucleus of mass number $A$ follow approximately the formula

$$n_A = \frac{1}{2} (\nu + 1) \bar{n}_H = \frac{1}{2} \bar{w} \bar{n}_H,$$

where $\bar{n}_H$ is the average multiplicity in nucleon-nucleon collisions, and $\bar{\nu} (\bar{w})$ is the average number of collisions (of wounded nucleons).

This formula suggests that the incident nucleon contribution to $\bar{n}_A$ is the same as the contribution of each hit target nucleon and equals *approximately* $\frac{1}{2} \bar{n}_H$. Thus, there seems to be no difference whether a nucleon is hit once or several times. This observation justifies the relevance of $w$.

For nucleus-nucleus collisions this picture implies that the average multiplicity in a collision of two nuclei with the mass numbers $A$ and $B$ is

$$\bar{n}_{AB} = \frac{1}{2} \bar{w} \bar{n}_H,$$  

while for nucleon-nucleus collisions one could also use $\bar{\nu}$ (as indicated in eq. (1.2)) here it is no longer possible.

The main purpose of this paper is to explore the consequences of the model in which the multiplicity distributions for nucleus-nucleus collisions are given by incoherent superposition of distributions provided by each wounded nucleon. We calculated average multiplicity and dispersion in this model. We found that the expected nuclear effects are rather dramatic, particularly for collisions of two heavy nuclei. Consequently, we feel that experimental investigation of heavy nuclei collisions at

* It may appear that this argument depends critically on the accuracy of eq. (1.2). We show later that this is not the case (see sect. 4).
high energies may indeed be useful for establishing: (a) which aspects of elementary collisions get amplified in nuclear interactions and (b) whether collective phenomena play an important role in nuclear collisions at high energies.

The collisions of two nuclei were already discussed by many authors. The extension of the Glauber model was used to describe elastic, quasi-elastic and total cross sections [9,10]. Production processes were also discussed in this framework [10].

Average multiplicities are discussed in sect. 2 and dispersion in sect. 3. In sect. 4 we consider the stability of the obtained results with respect to variation of the parameters of the model. Our conclusions are listed in the last section. The derivation of the formulae for the average number of wounded nucleons is given in appendix A. The dispersion of the number of wounded nucleons and of the number of collisions is derived in appendix B.

2. Average multiplicities

As explained in the introduction the average number of particles produced in inelastic nucleus (mass number $A$) — nucleus (mass number $B$) collision is

$$\bar{n}_{AB} = \frac{1}{2} \bar{w} \bar{n}_H.$$  \hspace{1cm} (2.1)

Thus calculation of $\bar{n}_{AB}$ reduces to calculating the average number of wounded nucleons $\bar{w}$. Since our basic assumption is that the inelastic collisions of two nuclei can be described as an incoherent composition of individual nucleons, we compute $\bar{w}$ using probability calculus.

In appendix A we show that the number of wounded nucleons in the collision of
A and B is the sum of wounded nucleons in the nucleus A and the nucleus B:

$$\bar{w}_{AB} = \bar{w}_A + \bar{w}_B,$$

where

$$\bar{w}_A = \frac{A\sigma_B}{\sigma_{AB}} \quad \text{and} \quad \bar{w}_B = \frac{B\sigma_A}{\sigma_{AB}}.$$  

Here $$\sigma_A$$ is the nucleon-nucleus A production cross section, $$\sigma_B$$ is the nucleon-nucleus B production cross section and $$\sigma_{AB}$$ is the production cross section for the collision of nucleus A with nucleus B. The explicit formulae for $$\sigma_A$$, $$\sigma_B$$ and $$\sigma_{AB}$$ are given in appendix A.

When $$B = 1$$, (2.2) reduces to

$$\bar{w}_{1A} = \frac{1}{\sigma_A} (A\sigma_H + \sigma_A) = 1 + \frac{A\sigma_H}{\sigma_A} = 1 + \bar{\nu},$$

in accordance with eq. (1.2). There, we have used the well-known expression for the average number of collisions $$\bar{\nu} = A\sigma_H/\sigma_A$$ [5], [11], where $$\sigma_H$$ is the nucleon-nucleon production cross section.

In fig. 2 we show $$R = \frac{1}{2} \bar{w}_{AB}$$ for various A and B nuclei and $$\sigma_H = 30$$ mb. It is seen that in first approximation $$R$$ is a function of the product $$AB$$. Furthermore, $$R$$

![Fig. 2. Average number of wounded nucleons (black symbols) and average number of collisions (open symbols) versus AB. $$\sigma_H = 30$$ mb.](image-url)
increases rather rapidly for $AB$ greater than $\sim 400$ (this is far beyond the region attainable in hadron-nucleus collisions). For comparison we also plot points for $R_\nu = \frac{1}{2}(\bar{R} + 1)$ which fit nicely hadron-nucleus collisions. The following nuclear densities were used:

For $A > 16$:

$$\rho(r) = \rho_0 \left(1 + \exp \left[\frac{r - R}{c}\right]\right)^{-1},$$

(2.5)

where $R = 1.07 A^{1/3}$ fm, $c = 0.545$ fm.

For $A = 4$:

$$\rho(r) = \left(\frac{1}{\pi R^2}\right)^{3/2} \exp\left(-\frac{r^2}{R^2}\right), \quad R = 1.37 \text{ fm}.$$ (2.6)

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Fig. 3. Dispersion versus average for (a) number of wounded nucleons (black symbols) and (b) number of collisions (open symbols). $\sigma_H = 30$ mb.
3. Dispersions

In our model each wounded nucleon contributes independently to the observed multiplicity distribution. Consequently, for a fixed number of wounded nucleons, the dispersion is given by the formula

$$D^2(w) = w \frac{1}{2} D^2_H,$$  \hspace{1cm} (3.1)

where $D^2_H$ is the dispersion in the nucleon-nucleon interaction. Thus, the observed dispersion can be computed from the formula

$$D^2 = \frac{1}{2} w D^2_H + \frac{1}{4} \left[ \bar{w}^2 - \bar{w}^2 \right] \bar{n}^2_H.$$  \hspace{1cm} (3.2)

Here we employed the following expression, implied by our model, for average multiplicity with a given number of wounded nucleons:

$$\bar{n}(w) = \frac{1}{2} w \bar{n}_H.$$  \hspace{1cm} (3.3)

Hence the problem of computing $D$ reduces to evaluation of $w^2 - \bar{w}^2$. This calculation is given in appendix A. The numerical results are shown in fig. 3 where dispersion of the number of wounded nucleons $w$ is plotted versus the average. The striking feature is that the observed dependence is approximately linear with the slope $\sim 1$. Similar linear behaviour is seen for dispersion of the number of collisions, but its slope is greater ($\sim 1.3$).

Fig. 4. Dispersion of multiplicity distribution versus average multiplicity for nucleus-nucleus collisions at lab energy 300 GeV/nucleon. $\sigma_H = 30$ mb, $\beta = 0.5$. 
In fig. 4 we plot $D_{AB}$ versus $\bar{n}_{AB}$ for 300 GeV obtained from eq. (3.2) using $\bar{n}_H = 8.5$ and $\bar{n}_H/D_H = 2.0$. Note that points corresponding to the hadron-nucleus collisions are all concentrated in the lower left corner of the diagram. This illustrates the extent of the extrapolation involved in our model.

4. Stability of the results

In this section we discuss two possible modifications of our model which are suggested by the analyses of the data for nucleon-nucleus collisions in refs. [1, 3]. We shall show that although they may be important in the case of nucleon-nucleus interactions the nucleus-nucleus collisions are far less sensitive to them.

The first observation is that all modifications of our model must preserve symmetry of the process of the nucleon-nucleon interaction. The simplest realization of such symmetry is to assume that both nucleons contribute $n_H$ of the multiplicity, as expressed in eq. (1.3). However, as discussed in refs. [1] and [3] reasonable fits to the data on nucleon-nucleus collision may be obtained with the formula

$$n_A = (\beta \bar{\bar{n}}_A + 1 - \beta) \bar{n}_H = (\beta \bar{\bar{w}}_A + 1 - 2\beta) \bar{n}_H,$$  \hspace{1cm} (4.1)

where $\beta < \frac{1}{2}$. Now it is clear that $\beta < \frac{1}{2}$ means that only $2\beta$ fraction of $\bar{n}_H$ gets multiplied in the collision whereas $(1 - 2\beta)$ does not. It is obvious that such a modification changes the predicted average multiplicity (since it is proportional to $\beta$). However, the relation between dispersion and average multiplicity seems to be unaffected by this modification of the model. We calculated dispersions for nucleus-nucleus collisions using eq. (4.1) (instead of eq. (1.3)) with $\beta = 0.4$. We found that such corrections introduce insignificant changes to the relation between $D$ and $\bar{n}_A$ (plotted in figs. 3 and 4) wherever the average number of wounded nucleons exceeds $\sim 10$, that means in the majority of cases.

Another possible modification is to change eq. (3.2). For example in ref. [3] the Poisson multiplicity distribution is assumed for nucleon-nucleon interactions. With this assumption the formula (3.2) should be replaced by

$$D^2 = \frac{1}{2} \bar{w} \bar{n}_H + \frac{1}{4}(\bar{w}^2 - \bar{\bar{w}}^2) \bar{n}_H^2,$$  \hspace{1cm} (4.2)

We have calculated dispersions with this formula and again found the modifications insignificant for nucleus-nucleus collisions. This is easily understood if one realizes that the first term in (4.2) is only a small correction to the leading second term provided the number of wounded nucleons exceeds $\sim 10$.

Thus other reasonable modifications of eq. (3.2) should also not change our conclusion for nucleus-nucleus interactions.
5. Conclusions

The standard description of hadron-nucleus interactions, in which inelastic collisions are incoherent compositions of the collisions of the incident hadron with individual nucleons in the target nucleus, is extrapolated to nucleus-nucleus collisions. The average multiplicities are sensitive to the details of the elementary nucleon-nucleon process and therefore can be used to fix the parameters of the model. On the other hand the relation between dispersion and average multiplicity depends almost exclusively on the mechanism of the amplification of the production process in consecutive collisions. It can be used therefore to distinguish between different mechanisms of multiplication of particles.

Appendix A

Formula for the average number of wounded nucleons.

The number of wounded nucleons in the collision of A and B is the sum of wounded nucleons in the nucleus A and the nucleus B. Thus it is enough to compute the average number of wounded nucleons in one nucleus e.g. B. Let us denote the probability of collision of the nucleon i from B with anyone of the nucleons of A with a given configuration $s_1^A, \ldots, s_A^A$ by

$$p(s_i^B; A; s_1^A, \ldots, s_A^A) \equiv p_A(s_i^B),$$

where the variables are defined in fig. 1. The probability that the nucleons $s_{i_1}^B, \ldots, s_{i_w}^B$ collide and $s_{i_{w+1}}^B, \ldots, s_{i_B}^B$ do not is

$$p_A(s_{i_1}^B) \ldots p_A(s_{i_w}^B) \cdot [1 - p_A(s_{i_{w+1}}^B)] \ldots [1 - p_A(s_{i_B}^B)].$$

After integrating over the configurations of B we get for the probability of having $w_B$ wounded nucleons in B:

$$P(w_B; B; A; s_1^A, \ldots, s_A^A; b)$$

$$= \binom{B}{w_B} [1 - \bar{p}(A; s_1^A, \ldots, s_A^A; b)]^{B - w_B}$$

$$\times [\bar{p}(A; s_1^A, \ldots, s_A^A; b)]^{w_B}.\quad(A.2)$$

Here

$$\bar{p}(A; s_1^A, \ldots, s_A^A; b) = \int d^2 s_i^B p(s_i^B; A; s_1^A, \ldots, s_A^A) D_B(b - s_i^B)\quad(A.3)$$
with

\[ D_B(s) = \int_{-\infty}^{+\infty} dz \, \rho_B(s, z), \]

where \( \rho_B(s, z) \) is the single nucleon probability distribution in the nucleus \( B \) (normalized to unity) which can be identified with the probability density from the single particle wave function. We assume all nucleons to be "equivalent" in the sense that all the one-nucleon probabilities are the same (this simplifying assumption can be removed at the expense of complicating, inessentially, the algebra which we want to avoid).

We are interested only in production processes, hence we should subtract the probability that none of the nucleons got wounded \( [P(w_B = 0)] \). Therefore, we normalize our probabilities as follows:

\[
\text{Norm} = \int d^2b [1 - P(w_B = 0)]
\]

\[
= \int d^2b [1 - \int d^2s^A_1 \ldots d^2s^A_{A'} d^2s^B_1 \ldots d^2s^B_{B'}
\times D_A(s^A_1) \ldots D_A(s^A_{A'}) D_B(s^B_1 - b) \ldots D_B(s^B_{B'} - b)
\times \prod_{i=1}^{B} \{1 - p(s^B_i; A; s^A_1, \ldots s^A_{A'})\}] = \sigma_{AB}, \tag{A.4}
\]

which is, in fact, just a cross section for production in a collision of the nucleus \( A \) with the nucleus \( B \) (for more details see the footnote in appendix B).

So, the average number of wounded nucleons in \( B \) is

\[
\sigma_{AB} \bar{w}_B = \int d^2b \, d^2s^A_1 \ldots d^2s^A_{A'}
\times \bar{w}_B(B, A; s^A_1, \ldots s^A_{A'}; b) D_A(s^A_1) \ldots D_A(s^A_{A'}), \tag{A.5}
\]

where

\[
\bar{w}_B(B, A; s^A_1, \ldots s^A_{A'}; b) = \sum_{w_B} w_B P(w_B, B, A, s^A_1, \ldots s^A_{A'}; b)
\]

\[
= \sum_{w_B=1}^{B} w_B \left( \frac{B}{w_B} \right) (1 - \bar{p})^{B - w_B} \bar{p}^{w_B} = B \bar{p}(A; s^A_1, \ldots s^A_{A'}; b). \]
So,

\[ \sigma_{AB} \bar{w}_B = B \int d^2 b \, d^2 s_A^1 \cdots d^2 s_A^A d^2 s_B^B \]

\[ \times D_A(s_A^1) \cdots D_A(s_A^A) D_B(s_B^B - b) \, p(s_B^B; A; s_A^1, \ldots s_A^A). \quad (A.6) \]

However,

\[ p(s_B^B; A; s_A^1, \ldots s_A^A) = 1 - \prod_{i=1}^A \left[ 1 - \sigma(s_B^B - s_i^A) \right], \quad (A.7) \]

where \( \int d^2 s^A \sigma(s) = \sigma_H \), \( \sigma_H \) being the nucleon-nucleon total inelastic cross section with diffraction production excluded. From (A.6) we get finally

\[ \bar{w}_B = \frac{B \sigma_A}{\sigma_{AB}}, \quad (A.8) \]

where \( \sigma_A \), the total inelastic nucleon-nucleus cross section, is

\[ \sigma_A = \int d^2 b \, d^2 s_B^B D_B(s_B^B - b) \left\{ 1 - \prod_{i=1}^A \left[ 1 - \int d^2 s_i^A D_A(s_i^A) \sigma(s_B^B - s_i^A) \right] \right\} \]

\[ = \int d^2 s_B^B \left\{ 1 - \prod_{i=1}^A \left[ 1 - \int d^2 s_i^A D_A(s_i^A) \sigma(s_B^B - s_i^A) \right] \right\} \quad (A.9) \]

because \( \int d^2 b \, D_B(s_B^B - b) = 1 \) for all \( s_B^B \). Repeating the same calculation for \( A \) we get

\[ \bar{w}_A = A \sigma_B / \sigma_{AB} \]

and the complete expression for the number of wounded nucleons is thus

\[ \bar{w}_{AB} = \frac{1}{\sigma_{AB}} \left( A \sigma_B + B \sigma_A \right). \quad (A.10) \]

Appendix B

Generating function for multiplicity distributions

All averages discussed in this paper can be obtained from the following two generating functions:

\[ F(x_1, \ldots x_A; y_1, \ldots y_B) = \prod_{i=1}^A \prod_{j=1}^B \left( 1 - \sigma_{ij} + x_i y_j \sigma_{ij} \right) \quad (B.1) \]
generates all averages where the number of wounded nucleons is relevant, and
\[
\phi(x) = \prod_{i=1}^{A} \prod_{j=1}^{B} (1 - \sigma_{ij} + x\sigma_{ij}) \quad (B.2)
\]
generates all averages where the number of collisions is relevant.

In (B.1) and (B.2)
\[
\sigma_{ij} = \alpha(b - s_i^A + s_j^B), \quad (B.3)
\]
where \(\alpha(s)\) is normalized, as always, to \(\sigma_H\):
\[
\int d^2s \sigma(s) = \sigma_H.
\]

Let us compute \(D^2_{AB}\) from (B.1). From (3.2) we know that the problem reduces to computing
\[
\overline{w^2 - \bar{w}^2} = (\overline{w_A^2} - \bar{w}_A^2) + (\overline{w_B^2} - \bar{w}_B^2) + 2\overline{w_A w_B} - 2\overline{w_A} \overline{w_B}. \quad (B.4)
\]
The first two terms in (B.4) one obtains from immediate generalizations of (A.5). For instance we have
\[
\sigma_{AB} w_B(w_B - 1) = B(B - 1) \int d^2b \ d^2s^A \ ... \ d^2s^A d^2s^B d^2s^B \ D_A(s_1^A) ... D_A(s_A^A) \ D_B(s_1^B) ... D_B(s_B^B)
\]
\[
= B(B - 1) \int d^2b \ d^2s^A \ ... \ d^2s^A d^2s^B d^2s^B \ \times \ p(s_1^B, A; s_1^A, ... s_A^A) p(s_2^B, A; s_1^A, ... s_A^A) D_A(s_1^A) ... D_A(s_A^A) \ D_B(s_1^B) ... D_B(s_B^B). \quad (B.5)
\]
Using the identity
\[
p(s_1^B, ... p(s_2^B ... = p(s_1^B ... + p(s_2^B ... - [p(s_1^B ... + p(s_2^B ... - p(s_1^B ... \cdot p(s_2^B ...),
\]
we obtain
\[
\sigma_{AB} w_B(w_B - 1) = B(B - 1) (2\sigma_A - \sigma_{2A}) \quad (B.6)
\]
and, mutatis mutandis,
\[
\sigma_{AB} w_A(w_A - 1) = A(A - 1) (2\sigma_B - \sigma_{2B}). \quad (B.7)
\]
Here \(\sigma_A(\sigma_B)\) is the nucleon-nucleus \(A(B)\) cross section and \(\sigma_{2A}(\sigma_{2B})\) is a cross section of a two-nucleon object with the spatial shape of the \(B(A)\) nucleus: \(D_B(s) [D_A(s)]\)
is its density distribution. So,

\[
\bar{w}_A^2 - \bar{w}_A^2 = \frac{1}{\sigma_{AB}} \left[ A^2(2\sigma_B - \sigma_{2B}) + A(\sigma_{2B} - \sigma_B) \right] - \frac{A^2\sigma_B^2}{\sigma_{AB}^2},
\]

\[
\bar{w}_B^2 - \bar{w}_B^2 = \frac{1}{\sigma_{AB}} \left[ B^2(2\sigma_A - \sigma_{2A}) + B(\sigma_{2A} - \sigma_A) \right] - \frac{B^2\sigma_A^2}{\sigma_{AB}^2}.
\]

(B.8)

We evaluate \(\bar{w}_A \bar{w}_B\) using the generating function (B.1). Expanding \(F\) in powers of \(x_1, \ldots x_A, y_1, \ldots y_B\) we find that the coefficients of various products give probabilities of all possible collisions. For instance the coefficient of \(x_1^3 x_2^3 y_1^2 y_2^2\) term is the probability of a 6-fold collision in which two nucleons from \(A\) collided three times each and three nucleons from \(B\) collided twice each, hence 5 nucleons got wounded. So the number of wounded nucleons given by one term is equal to the number of different \(x_i's\) (\(w_A\)) and the number of different \(y_k's\) (\(w_B\)).

Let us denote

\[
F(1, \ldots 1; y_1, \ldots, y_B) = F(1, y),
\]

\[
F(1, \ldots, x_i = 0, \ldots 1; y_1, \ldots, y_B) = F_i(0, y),
\]

\[
F(1, \ldots, x_i = 0, \ldots 1; 1, \ldots, y_k = 0, \ldots 1) = F_{ik}(0, 0).
\]

(B.9)

\(F(1, y)\) contains all the probabilities of \(F\), while \(F_i(0, y)\) contains all the probabilities of \(F\) except the ones with \(x_i\) to any power. Therefore \(G(y) = F(1, y) - F_i(0, y)\) contains only these probabilities which are multiplied by \(x_i\) to any power and consequently \(H(y) = \sum_i G_i(y)\) is the sum of all the probabilities with weights \(w_A\) and still multiplied by \(y_k's\). But we want to have all the probabilities summed with weights \(w_A w_B\). The following expression is the one we want

\[
I = \sum_k \left[ H(y_1 = 1, \ldots y_B = 1) - H(y_1 = 1, \ldots y_k = 0, \ldots y_B = 1) \right]
\]

\[
= ABF(1, 1) - B \sum_{i=1}^{A} F_i(0, 1) - A \sum_{k=1}^{B} F_k(1, 0) + \sum_{i=1}^{A} \sum_{k=1}^{B} F_{ik}(0, 0),
\]

(B.10)

where we have re-traced our steps back to the original generating function \(F\) [(B.9) explains the notation].

\[
\sigma_{AB} \bar{w}_A \bar{w}_B = \int d^2 b \int d^2 s_A^1 \ldots d^2 s_A^A d^2 s_B^1 \ldots d^2 s_B^B f(b, s_A^1, \ldots s_A^A) D_A(s_A^1) \ldots D_A(s_A^A) D_B(s_B^1) \ldots D_B(s_B^B).
\]

(B.11)
We normalize the probabilities similarly as in sect. 2. The expression for the norm symmetric in $A$ and $B$ variables is therefore

\[
\text{Norm} = \sigma_{AB} = \int d^2b \int d^2s^A_1 \ldots d^2s^A_1 \ldots d^2s^B_B \times D_A(s^A_1) \ldots D_A(s^A_1) D_B(s^B_1) \ldots D_B(s^B_1) \{1 - \prod_{i=1}^A \prod_{j=1}^B [1 - \alpha(b - s_i + s_j)]\} .
\] (B.12)

From (B.1) we get

\[
F(1,1) = 1, \quad F_i(0,1) = \prod_{k=1}^B (1 - \alpha_{ik}) , \quad F_k(1,0) = \prod_{i=1}^A (1 - \alpha_{ki}) ,
\]

\[
F_{ik}(0,0) = \prod_{l \neq i}^A \prod_{j \neq k}^B (1 - \alpha_{lj}) .
\] (B.13)

The final result, after a straightforward algebra, is

\[
\sigma_{AB} \overline{w} A \overline{w} B = (\sigma_A + \sigma_B - S_{AB} - \alpha_H N_{A-1} N_{B-1}) A B
\] (B.14)

with $S_{AB}, N_{A-1}$ and $N_{B-1}$ defined as follows

\[
S_{AB} = \int d^2b \{1 - \int d^2s D_A(s)(1 - \alpha_H D_B(b - s))^{B-1} \} [\int d^2s' D_B(s')] \times (1 - \alpha_H D_A(b + s'))^{A-1} \}
\]

\[
N_{A-1} = \int d^2s D_A(s) [1 - \alpha_H D_A(s)]^{A-1} ,
\]

\[
N_{B-1} = \int d^2s D_B(s) [1 - \alpha_H D_B(s)]^{B-1} .
\] (B.15)

From (B.4), (B.8) and (B.15) we get the final formula

\[
\overline{w^2} - \overline{w^2} = \frac{1}{\sigma_{AB}} \left[ A^2(2\sigma_B - \sigma_{2B}) + A(\sigma_{2B} - \sigma_B) \right] - \frac{A^2 \sigma_B^2}{\sigma_{AB}^2}
\]
which enables us to compute $D_{AB}^2$ given by (3.2) *.

It is worth noticing that (B.16) reduces, in the case of the nucleon-nucleus $A$ collision, to a compact and handy expression:

$$\bar{w}_A^2 - \bar{w}_A^2 = \frac{A^2 \sigma_H^2}{\sigma_A^2} \left[ \sigma_A \int d^2b \, D^2(b) - 1 \right] + \frac{A \sigma_H}{\sigma_A} \left[ 1 - \sigma_H \int d^2b \, D^2(b) \right].$$

From the generating function (B.1) one can also obtain the formula (2.2) for the average number of wounded nucleons. To this end one employs the function $H(y = 1)$ which, after averaging over nuclear densities gives $\sigma_{AB} \bar{w}_A$.

The average number of collisions $\bar{v}$ and the dispersion $D^2 = \bar{v}^2 - \bar{v}^2$ we calculate from the generating function

$$\phi(x) = \prod_{i=1}^{A} \prod_{j=1}^{B} (1 - \sigma_{ij} + x \sigma_{ij}).$$

* In the actual numerical calculations we used the optical limit formula for $\sigma_{AB}$

$$\sigma_{AB} = d^2b \left[ 1 - \exp(-AB \sigma_H \int d^2s \, D_A(s) \, D_B(b - s)) \right]$$

and the following formula for $\sigma_A(\sigma_B)$, $\sigma_{2A}(\sigma_{2B})$:

$$\sigma_A = \int d^2b \left[ 1 - (1 - \sigma_H D_A(b))^A \right],$$

$$\sigma_{2A} = \int d^2b \int d^2b_1 d^2b_2 \, D_B(b - b_1) \, D_B(b - b_2) \left[ 1 - (1 - \sigma_H D_A(b_1) - \sigma_H D_A(b_2))^A \right]$$

$$- \sigma_H^2 A \left[ \int d^2b \, D_B(b)^2 \int d^2b \, (1 - 2 \sigma_H D_A(b))^A - 1 \right],$$

which can be derived from the exact multiple scattering formula under only one assumption that the nucleon size is small compared to sizes of nuclei $A$ and $B$. 
Let us label the pair of $ij$ with one index $p$, $1 \leq p \leq AB$, and re-write
\[ \phi(x) = \prod_{p=1}^{AB} (1 - \sigma_p + x\sigma_p) = \sum_{\nu} \prod_{p}^{AB-\nu} (1 - \sigma_p) \prod_{p'}^{\nu} (x\sigma_{p'}) , \]  
(B.19)
where $\Sigma$ extends over all possible divisions of the set of $AB$ indices into two groups.

\[ \frac{\partial \phi(x)}{\partial x} \bigg|_{x=1} = \sum_{\nu} \prod_{p}^{AB-\nu} (1 - \sigma_p) \prod_{p'}^{\nu} \sigma_{p'} = \sum_{p=1}^{AB} \sigma_p . \]  
(B.20)
This last expression is clearly the average number of collisions (each element of the first sum of (B.20) is a product of $\nu$, the probability that $AB - \nu$ nucleons did not collide and the probability that $\nu$ nucleons did collide) for a given configuration of the nucleons in the colliding nuclei. So, to obtain the average number of collisions for two given nuclei we have to average $\frac{\partial \phi(x)}{\partial x} |_{x=1}$ over the nuclear densities and divide by $\sigma_{AB}$:

\[ \bar{\nu} = \frac{AB \int d^2b d^2s^A d^2s^B D_A(s^A) \sigma(b - s^A + s^B) D_B(s^B)}{\sigma_{AB}} = \frac{AB \sigma_H}{\sigma_{AB}} . \]  
(B.21)
Using the same generating function (B.18) we obtain the dispersion as follows:

\[ \sigma_{AB}(\nu(\nu - 1)) = \sigma_{AB} \left( \frac{\partial^2 \phi(x)}{\partial x^2} \right) \bigg|_{x=1} = \sigma_{AB} \sum_{\nu} \sum_{p \neq p'}^{AB} \sigma_p \sigma_{p'} \]

\[ = \int d^2b d^2s^A_1 \ldots d^2s^B \sum_{i,k} s^A_i \ldots D_B(s^B) \sum_{i,k} \sigma(b - s^A_i + s^B_k) \]

\[ \times \sum_{l,n \neq i,k} \sigma(b - s^A_l + s^B_m) , \]  
(B.22)
where $l, n \neq i, k$ means that the two pairs are different. We have therefore the following three expressions to compute (we assume $\sigma(s) = \sigma_H \delta^2(s)$):

(i) \[ i \neq l, k = n , \]

\[ \int d^2b d^2s^A_1 \ldots d^2s^B \sum_{i,k} \sigma(b - s^A_i + s^B_k) \sum_{l \neq i} \sigma(b - s^A_l + s^B_k) \]

\[ = BA(A - 1) \sigma_H^2 \times \int d^2b D^2_A(b) , \]
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(ii) \[ i = l, k \neq n , \]
\[
\int d^2b \; d^2s_1^A \ldots d^2s_B^B \; D_A(s_1^A) \ldots D(s_B^B) \sum_{i,k} \sigma(b - s_i^A + s_k^B) \sum_{k \neq n} \sigma(b - s_i^A + s_n^B) \\
= AB(B - 1) \sigma_H^2 \int d^2b \; D^2_B(b) ,
\]

(iii) \[ i \neq l, k \neq n , \]
\[
\int d^2b \; d^2s_1^A \ldots d^2s_B^B \; D_A(s_1^A) \ldots D(s_B^B) \sum_{i,k} \sigma(b - s_i^A + s_k^B) \sum_{k \neq n} \sigma(b - s_i^A + s_n^B) \\
= AB(A - 1) (B - 1) \sigma_H^2 \int d^2b \; \chi^2(b) ,
\]

where \( \chi(b) = \int d^2s \; D_A(b + s) \; D_B(s) \). The final formula is:
\[
\sigma_{AB}D^2 = AB \sigma_H^2 \left\{ (A - 1) (B - 1) \int d^2b \; \chi^2(b) + (B - 1) \int d^2b \; D_B^2(b) \\
+ (A - 1) \int d^2b \; A^2_D(b) - \frac{AB}{\sigma_{AB}} + \frac{1}{\sigma_H} \right\} .
\] (B.23)

References

[1] K. Gottfried, 5th Int. Conf. on high-energy physics and nuclear structure, Uppsala 1973 and
published in Phys. Rev. D.